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Fairness and Efficiency in a Random Assignment: Three Impossibility Results*

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Abstract

This paper considers the problem of allocating N indivisible objects among N agents according to their preferences when transfers are not allowed, and studies the tradeoff between fairness and efficiency in the class of strategy-proof mechanisms. The main finding is that for strategy-proof mechanisms the following efficiency and fairness criteria are mutually incompatible: (1) Ex-post efficiency and envy-freeness, (2) ordinal efficiency and weak envy-freeness and (3) ordinal efficiency and equal division lower bound. Result (1) is the first impossibility result for this setting that uses ex-post efficiency; results (2) and (3) are more relevant for practical implementation than similar results in the literature. In addition, for $N = 3$ the paper strengthens the characterization result by Bogomolnaia and Moulin (2001): the random serial dictatorship mechanism is the unique strategy-proof, ex-post efficient mechanism that eliminates strict envy between agents with the same preferences.

JEL Classification: *C78; D71; D78*

Key words: random assignment; random serial dictatorship; strategy-proofness; ex-post efficiency; weak envy-freeness; equal division lower bound.

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1 Introduction

Optimal allocation of goods among individuals is one of the core issues of economics. Normally researchers address this issue using the well-established concepts of markets and auctions, in which individuals receive goods in exchange for transfers. However, in a variety of real-life situations these transfers are not available for either ethical, institutional or other reasons. Recent literature analyzes numerous examples of such situations: from student assignment to primary schools [Abdulkadiroğlu, Che and Yasuda (2011)] and job placement for graduates [Roth (1984), Coles et al. (2010)], to housing markets [Chen and Sönmez (2002)], organ donation [Roth, Sönmez and Ünver (2005)] and distributing military supplies [Kesten and Yazici (2012)].

Such circumstances have forced regulators to design new artificial *matching markets* and develop *assignment mechanisms* suitable for these markets. But no matter how diverse the matching markets might be, all mechanisms are usually required to have the same three qualities: to be *strategy-proof*, *efficient,* and *fair*. This paper studies the mutual compatibility of these three properties and focuses on the notions of fairness and efficiency that are most relevant for the real-life applications. It is also the first paper in the matching and random assignment literature to provide the impossibility result with ex-post efficiency.

The paper considers the class of strategy-proof mechanisms. Strategy-proofness is a term for incentive compatibility or robustness to manipulation. A mechanism is called strategy-proof if agents always prefer to report their preferences truthfully rather than attempt to strategically manipulate them. If, on the contrary, a mechanism is *not* strategy-proof, its outcome cannot be reliably predicted and the resulting allocation will not necessarily have any of the other expected properties. Therefore, strategy-proofness is usually considered to be a crucial property.

Perhaps the most straightforward and well-known strategy-proof mechanism is the *serial dictatorship mechanism (SD)*. It works as follows: all agents choose their objects sequentially according to some exogenous order. It is easy to see that this mechanism always results in an efficient allocation (in the Pareto sense). At the same time it is intuitively clear that SD is very *unfair*: the first agent picks any object, the second gets at least her second best, while the last agent has no choice. The natural "fair" extension of the SD mechanism is the dictatorship in which the underlying order is chosen *randomly.* The resulting mechanism, *random serial dictatorship (RSD)*, induces *only* efficient outcomes, therefore it is *ex-post efficien*t. Ex-post efficiency is a rather weak requirement. It is also, perhaps, the most applicable efficiency property for real-life situations as it is relevant for the final, deterministic outcomes. Because RSD is ex-post efficient and strategy-proof it has become one of the most widely used mechanisms both directly and indirectly, as a component of more complex mechanisms.

In the recent literature the central role of RSD among other mechanisms has been supported by several equivalence results that connect RSD to versions of other mechanisms used in practice. For example, Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to the *core from random* *endowments* mechanism, that initially randomly allocates objects and then proceeds by using the *top trading cycles (TTC)* algorithm in which agents voluntarily exchange the objects that they are endowed with. Next, Kesten (2009) shows the equivalence with another TTC-based mechanism, the *top trading cycles from equal division*. Finally, the celebrated *deferred acceptance* mechanism introduced by Gale and Shapley (1962), which is often used for the *two-sided matching problems* such as the school choice problem (as well as the college admission problem and job placement problem), is also equivalent to RSD in case schools are initially indifferent between students and the ties are broken randomly for all schools together.

Finally, the third trait a real-life mechanism is expected to possess apart from being strategy-proof and ex-post efficient, is to be *fair*. In addition to purely normative reasons, satisfying some form of fairness is important in order to prevent agents from contracting outside of the centralized market, which can possibly damage the entire assignment.[1] Since *ex-post fairness* is an extremely restrictive property[2] it is common to address fairness from the *ex-ante* perspective, which implies not a fair distribution of objects but a fair distribution of the *chances* to get these objects. Ex-ante fairness is therefore defined over the set of individual assignment probabilities, called *random assignments*.

One of the ex-ante fairness notions most commonly used in the literature is *envy-freeness*. A random assignment is envy-free if every agent prefers her own random assignment to anyone else's. Although this property should almost perfectly eliminate any fairness concerns, it also proves to be too restrictive. For instance, the RSD mechanism introduced above is not envy-free if there are three or more agents.

In this paper I show a general impossibility regarding envy-freeness: the set of envy-free, strategy-proof, and ex-post efficient mechanisms is empty. This result is most relevant for deterministic assignment mechanisms. Since ex-post efficiency is the minimal efficiency requirement for these mechanisms (and it can only be expected to occur if strategy-proofness is satisfied as well), this impossibility result implies a certain limit on the potentially reachable level of fairness. (Non-envy-freeness limit is certainly not binding for I show a stronger impossibility result in the Lemma 1 below.) The main example of a strategy-proof ex-post efficient mechanism is the mechanism based on the TTC algorithm that is used in school choice, organ donation, and housing problems (for details see Abdulkadiroğlu and Sönmez (2003), Roth et al. (2004) and Abdulkadiroğlu and Sönmez (2010) respectively). The theorem therefore shows that no mechanism that is based on the TTC algorithm (so that it remains strategy-proof and ex-post efficient) can be envy-free, regardless of the initial endowment structure and the randomization of these endowments.

The second part of the paper deals with a less strict fairness property: *weak envy-freeness*.

[1]For deterministic mechanisms in two-sided matching literature the central notion related to fairness is *stability*, which guaranties the impossibility of a pair of agents profitably contracting outside of the market. This requirement is incompatible with strategy-proofness and ex-post efficiency, similar to the results considered in this paper. See, for example, Abdulkadiroğlu and Sönmez (2003) for a discussion of this tradeoff for a school choice problem.

[2]See Kesten and Yazici (2010) for discussion of ex-post fair mechanisms.

Formally, a random assignment is weakly envy-free if for each agent her own assignment is not strictly stochastically dominated by some other agent's assignment. As an example, consider the following random assignment:

$$\begin{array}{c|ccc} & h_1 & h_2 & h_3 \\ \hline a_1 & .5 & .0 & .5 \\ a_2 & .3 & .4 & .3 \\ a_3 & .2 & .6 & .2 \end{array},$$

where all three agents a_1, a_2, a_3 prefer object h_1 to object h_2, and object h_2 to object h_3. Then this random assignment is weakly envy-free because none of the individual lotteries stochastically dominates another.

Weak envy-freeness is important for several reasons. First, if one agent does not weakly envy another, there always exist a set of Bernoulli utilities which are consistent with ordinal preferences for which envy-freeness is strict.[3] In the example above agent a_1, for instance, prefers her assignment to the other two whenever her preference for house h_1 is strong enough. Similarly, agent a_3 prefers her assignment whenever her dislike of house h_3 is strong enough.

Secondly, weak-envy-freeness becomes even stronger in real-life applications, as compared to the case of abstract rational agents, if we account for bounded rationality. To the best of my knowledge there is no relevant research on envy in the lab, but there is a vast related literature dealing with the so-called endowment effect, or the difference between the willingness to accept and the willingness to pay (WTA-WTP) for some goods. In these experiments agents value the goods that they are endowed with significantly more than the goods that they can purchase. This result holds in different settings and for different types of goods: lotteries over monetary outcomes, private goods such as coffee mugs and chocolate bars, and non-consumption goods such as decreased food risk and health insurance as well as public goods. There is, unfortunately, no WTA-WTP study for the case of *lotteries over non-consumption goods*, such as school slots, which would be the most relevant framework for the random assignment problem that we consider here. However, we can extrapolate the existing results: in their review of the WTA-WTP literature Horowitz and McConnell (2002) find that the average WTA/WTP ratio rises significantly from 2.10 (meaning that subjects are willing to sell a good for a price that is two times higher than the sum they are willing to pay for the same good) for the case of monetary lotteries to 10.41 for the case of non-consumption goods. This suggests the existence of a sizable WTA/WTP ratio for

[3]This, however, cannot always be translated for the case of an entire random assignment since different pairwise comparisons might require mutually incompatible utilities. In the same example above, if agent a_2 does not envy agent a_1, then she necessarily envies agent a_3. A random assignment for which such non-envy utilities exist is called *possibly envy-free*, which is stricter than weak-envy-free. This distinction is not very common in the random assignment literature since most of the known weak-envy-free mechanisms are also possible-envy-free. Moreover, since our focus is on negative results, we also concentrate on the lighter notion of weak-envy-freeness. For more detail on possible envy-freeness see Aziz et. al (2014).

the lotteries over non-consumption goods as well.[4] Therefore, one can expect that the agent is less likely to envy others due to the endowment effect because of the readjustment of her *cardinal* preferences ex-post − after being assigned a lottery. This readjustment can be relatively mild in the case of the weak-envy-free assignment because it would not necessarily involve any change in *ordinal* preferences. For instance, in the example above, the agent a_1 after being endowed with her lottery can value house h_1 somewhat higher (or house h_3 − somewhat lower) so that she does not envy other agents. However, if a random assignment is *not* weak-envy-free, then envy can be eliminated *only* if the agent changes her ordinal preferences since some other agent will have a stochastically dominant lottery. Clearly, this type of preference adjustment is stronger since it always implies a change of cardinal preferences and cannot be as robust as the endowment effect.

Finally, an agent who gets a non-weak-envy-free assignment can often claim to be treated unfairly. For instance, if agent a_2 in the example above preferred house h_2 to all other houses, then this assignment is not weak-envy-free. If a_2 did not get her most preferred house h_2 (which happens with a 60% probability), she might justifiably claim to have been treated worse than agent a_3 since she got a stochastically dominated lottery. Once there is a legal basis for a lawsuit of some type of discrimination, it can be based exclusively on the verifiable information (reported preferences and the assigned lotteries) and not on the agent's private information (as in the case of strict envy-freeness). Clearly, it is important for the mechanism designer to avoid such risks.

All in all, weak-envy-freeness appears to be a reasonable minimum fairness requirement for real-life mechanisms. The second result of this paper restricts, however, the set of feasible weakly envy-free mechanisms. To understand this restriction, we need another notion of efficiency: *ordinal efficiency*,[5] which implies the ex-post efficiency used in the first result. Ordinal efficiency was first introduced by Bogomolnaia and Moulin (2001), hereinafter referred to as BM, in their seminal paper in which they demonstrate that although RSD is efficient ex-post, it can lead to systematic inefficiencies ex-ante. In contrast, an ordinally efficient mechanism never induces random assignments that are (first-order) stochastically dominated by some other random assignment.

In this paper I show that there does not exist a mechanism which is weak-envy-free, strategy-proof, and ordinally efficient. This result is most strongly related to the impossibility result in BM, that is, the mutual incompatibility of strategy-proofness, ordinal efficiency, and equal treatment of equals (or ETE, which requires that agents with identical preferences receive identical individual random assignments).[6] ETE and weak-envy-freeness are logically independent, but the latter applies to a much larger set of preference profiles (all possible profiles as compared to those which contain identical preferences) and is arguably more relevant for real-life applications for the reasons listed above.

[4]In a more recent study Isoni, Loomes and Sugden (2011) show that the WTA/WTP disparity is more robust for monetary lotteries than for coffee mugs even in settings specifically designed to neutralize slight misconceptions of agents.

[5]Ordinal efficiency is often called sd-efficiency for (first-order) stochastic dominance.

[6]Zhou (1990) shows a similar result for the case of three agents but for the cardinal preference domain.

The last impossibility result of the paper states that there is no strategy-proof and ordinally efficient mechanism such that its random assignments are preferred to the random uniform lottery by all agents. The latter property is called *equal division lower bound (EDLB)* and it can be seen as an individual rationality constraint if we assume that each agent is initially entitled to the equal share $\frac{1}{N}$ of each object and can deviate to this option in case she does not like her individual assignment. In other words, if the problem includes the outside option of equal division, then every strategy-proof mechanism is necessarily ordinally inefficient (and thus ex-ante inefficient).

From a practical point of view, equal division lower bound appears to be important for two main reasons. First of all, equal division seems to be the most natural fair assignment and thus a natural benchmark to compare all other random assignments to.[7] Secondly, equal division is often used in practice − whenever the assignment is made in the absence of or regardless of the data on agents' preferences, for instance. This is the case in the process of assigning Japanese teachers to Japanese schools abroad (Nihonjin gakkō). Each successful applicant is sent for two to three years to one of more than 80 schools all over the world regardless of his or her actual preferences.

From a theoretical point of view, equal division lower bound is related more to efficiency than to fairness, as compared to weak envy-freeness and ETE. Unlike the other two notions, EDLB does not compare the individual assignments to each other but to the (almost always inefficient) equal division benchmark. Therefore, EDLB does not require the assignment to be fair in the egalitarian sense, but only that this assignment *dominates* the most egalitarian assignment − equal division.

Besides RSD, another famous solution to the random assignment problem that satisfies equal division lower bound was offered by Hylland and Zeckhauser (1979). Their mechanism uses the concept of competitive equilibrium with equal incomes (CEEI) to fairly divide the probabilities of the objects: agents are endowed with equal probability shares and trade them against each other at market prices. CEEI induces efficient and envy-free random assignments but it is not strategy-proof, which follows from both Zhou (1990) and BM. However, the question remains whether one can construct a strategy-proof mechanism that would, similarly to CEEI, start with equal division and eventually arrive at an efficient random assignment by mutual exchange of probability shares. This random assignment does not need to (and cannot) satisfy the equal treatment of equals or weak envy-freeness, but would necessarily satisfy equal division lower bound. Unfortunately, according to the impossibility theorem, such a mechanism does not exist.

Despite the negative results presented in this paper we, however, can still hope to find a strategy-proof, fair, and efficient mechanism in some relevant cases. For large markets in which every object has an increasing number of copies (for example, one can think of slots in one school as copies of a unique slot), Che and Kojima (2010) show that RSD is asymptotically ordinally efficient. For a similar large market Kojima and Manea (2010) show that the envy-free and ordinally efficient

[7] An extensive review on comparison to equal division and other notions of fairness for allocation rules is made by Thomson (2007).

Table 1: Summary of results

		Strategy-proof mechanisms			
		Envy-free	Weak envy-free	Equal division lower bound	Equal treatment of equals
Ex-post efficient	$N = 3$	$\emptyset$(Theorem 1, BM*)	RSD! (Corollary 2)	RSD	RSD! (Corollary 1, BM)
	$N > 3$	$\emptyset$ (Theorem 1)	RSD (BM)	RSD	RSD
Ordinally efficient	$N > 3$	$\emptyset$	$\emptyset$ (Theorem 2)	$\emptyset$ (Theorem 3)	$\emptyset$ (BM)

Exclamation mark denotes uniqueness, BM stands for Bogomolnaia and Moulin (2001).
*The case of three agents is also mentioned by BM, p.310, though informally.

probabilistic serial mechanism studied by BM is also asymptotically strategy-proof.[8] Therefore, the impossibility results presented here do not hold asymptotically for these types of large markets.

Finally, for the case of three agents I characterize RSD as a unique mechanism that is strategy-proof, ex-post efficient, and that eliminates strict envy between agents with identical preferences (I call the latter property *weak envy-freeness for equals*). This result also implies that a mechanism is strategy-proof, ex-post efficient, and weak envy-free if and only if it is RSD. Similarly, it also implies the characterization of RSD in BM, in which the authors use the equal treatment of equals instead of weak-envy-freeness.

Weak envy-freeness for equals can be seen as a natural relaxation of both the equal treatment of equals and the weak envy-freeness. On the one hand, weak envy-freeness for equals does not restrict individual random assignments for two agents with identical preferences *unless* one of them strictly envies another. In contrast, equal treatment of equals puts its restriction even in cases where no agent necessarily envies someone else and such a restriction can be redundant. On the other hand, weak envy-freeness for equals does not restrict individual random assignments of two agents if one of them strictly envies another *unless* they have identical preferences, while weak envy-freeness does so as if the mechanism designer must guarantee equitable treatment even for different agents, which can also be seen as a redundantly strict constraint.

Table 1 summarizes the main findings of this paper as well as the relevant results of BM.

The paper proceeds as follows: Section 2 introduces the framework, section 3 presents the first impossibility result (Theorem 1), section 4 covers the characterization result (Proposition 1) and the second impossibility result (Theorem 2), section 5 presents the third impossibility result (Theorem 3), and section 5 concludes by discussing the implications and the limitations of the findings.

[8]Based on the probabilistic serial mechanism Budish et al. (2013) develop fair and efficient mechanisms for various non-standard settings.

2 The Model

In this section I introduce the framework: define the house allocation problem, the random assignment mechanism and its properties.

Let $A = \{a_1, a_2, ..., a_N\}$ be the set of N agents and $H = \{h_1, h_2, ..., h_N\}$ be the set of N houses. Each agent $a \in A$ is endowed with a strict preference relation $\succ_a$ on H with a corresponding weak preference relation $\succcurlyeq_a$. A set of individual preferences of all agents constitutes a *preference profile* $\succ = (\succ_a)_{a\in A}$. Let $\mathcal{R}$ be the set of all possible individual preferences, and $\mathcal{R}^N$ be the set of all possible preference profiles. In what follows we assume that the sets A and H are fixed and that the house allocation problem is defined by the preference profile $\succ$ only.

Each house allocation problem has either a deterministic solution called matching or a probabilistic solution called random assignment. A *random assignment* P is a bistochastic matrix (with a sum of elements in any row and any column being equal to one) of size N containing non-negative elements. Each element $P_{a,h}$ of the matrix P represents a probability of agent a being assigned house h. Let $\mathcal{P}$ be a set of all possible random assignments P. A *matching* μ is a random assignment whose elements can only be zeros or ones, so that μ precisely prescribes which agent receives which house. Let $\mathcal{M}$ be a set of all possible matchings μ. According to the Birkhoff-von Neumann theorem any random assignment P can be represented as a lottery over the set of matchings $\mathcal{M}$ (but this representation is not necessarily unique). For this reason and since agents care only about their own assignment, we can concentrate on random assignments without specifying the exact matchings these random assignments correspond to.

In order to compare different random assignments we need the following definitions. A set of houses that agent a weakly prefers to some house h is the *upper contour set of house h at $\succ_a$*: $U(\succ_a, h) = \{h' \in H : h' \succcurlyeq_a h\}$. Given the individual random assignment P_a the overall probability of agent a being assigned some house that is at least as good as house h is the *surplus at h under P_a*: $F(\succ_a, h, P_a) = \sum_{h'\in U(\succ_a,h)} P_{a,h'}$. An individual random assignment P_a *(first order) stochastically dominates* another individual random assignment P'_a at $\succ_a$ (denoted by $P_a \geq_a P'_a$) if all surpluses of the former weakly exceed the surpluses of the latter: for each $h \in H$ $F(\succ_a, h, P_a) \geq F(\succ_a, h, P'_a)$. A *strict domination* ($P_a >_a P'_a$) occurs under the additional condition that the two random assignments are not equal: $P_a >_a P'_a \iff (P_a \geq_a P'_a) \wedge (P_a \neq P'_a)$. Finally, a random assignment P is said to *(strictly) dominate* another random assignment P' if it dominates for all agents individually (and dominates strictly at least for one agent).

2.1 Properties of mechanisms

From here on we deal with systematic procedures called *mechanisms* that associate each preference profile $\succ \in \mathcal{R}^N$ with a random assignment $P \in \mathcal{P}$: $P = \varphi(\succ)$, where φ denotes a mechanism.

Efficiency. One of the most important properties in the house allocation problem is efficiency.

A matching is *efficient* at some preference profile if it is not dominated by any other matching at this preference profile. A random assignment is said to be *ex-post efficient (ExPE)* at a preference profile if it can be represented as a lottery over matchings that are efficient at this preference profile. If a random assignment is not dominated by any other random assignments, it is said to be *ordinally efficient (OE)*. Similarly, a mechanism is said to be *ex-post efficient (ordinal efficient)* if for any preference profile it results in an ex-post efficient (ordinally efficient) random assignment.

Strategy-proofness. Another important type of property which a mechanism is desired to satisfy is strategy-proofness. A mechanism φ is *strategy-proof (SP)* if at any preference profile no agent can benefit by misreporting her preferences: for each $a \in A$, for each $\succ \in \mathcal{R}^N$ and for each $\succ'_a \in \mathcal{R}$ the following holds: $\varphi(\succ) \geq_a \varphi_a(\succ'_a, \succ_{-a})$. A mechanism φ is said to be *weakly strategy-proof (wSP)* if at any preference profile no agent can get a strictly stochastically dominating assignment by misreporting her preferences: for each $a \in A$ and each $\succ \in \mathcal{R}^N$ there does not exist $\succ'_a \in \mathcal{R}$ such that $\varphi_a(\succ'_a, \succ_{-a}) >_a \varphi_a(\succ)$.

Now I introduce an auxiliary notion of strategy-proofness that is used for the impossibility result below. A mechanism is said to be *upper-shuffle-proof (USP)* if by shuffling some of the top choices in her reported preferences an agent does not change her corresponding surplus (although she might still benefit from using other strategies): for each $a \in A, h \in H$, and for each $\succ \in \mathcal{R}^N, \succ'_a \in \mathcal{R}$ such that $U(\succ_a, h) = U(\succ'_a, h)$, the following holds: $F(\succ_a, h, \varphi_a(\succ)) - \varphi_{ah}(\succ) = F(\succ_a, h, \varphi_a(\succ'$ $)) - \varphi_{ah}(\succ')$ (the difference represents the sum of assignment probabilities for houses that are strictly better than h). Note that upper-shuffle-proofness is weaker than strategy-proofness, and neither implies nor is implied by weak strategy-proofness since it has stronger implications than weak strategy-proofness but applies to a smaller set of preference profiles. [9]

Fairness. Finally, the third type of desirable property of a mechanism is fairness. A random assignment P is *envy-free (EF)* if every agent prefers her assignment to any other agent's assignment: for each $a, a' \in A$ $P_a \geq_a P_{a'}$. A random assignment P is *weakly envy-free (wEF)* if no agent strictly prefers some other agent's assignment: there do not exist $a, a' \in A$ such that $P_{a'} >_a P_a$. Another widely used notion of fairness is the *equal treatment of equals (ETE)*: for each $a, a' \in A$ with $\succ_a = \succ_{a'}$ the individual random assignments are identical: $P_a = P_{a'}$. The third established fairness notion that we use is the *equal division lower bound (EDLB)*. A random assignment satisfies the equal division lower bound if each agent weakly prefers her individual assignment to the fair division assignment. Finally, a mechanism is said to be envy-free (weak envy-free; to satisfy the equal treatment of equals; to satisfy the equal division lower bound) if it *always* results in random assignments that are envy-free (weak envy-free; satisfy equal treatment of equals; satisfy equal division lower bound). Note that weak envy-freeness and equal division lower bound apply to all preference profiles but has a mild implication for a random assignment, whereas ETE applies only to specific preference profiles and has strict implications. For this reason neither ETE nor wEF, or EDLB imply one another, although all of them follow from envy freeness.

[9]Upper-shuffle-proofness is the same as lower invariance in Mennle and Seuken (2014).

Next I introduce two auxiliary notions of fairness: upper envy-freeness and the strong equal treatment of equals. A random assignment P is *upper envy-free (UEF)* if any two agents with identical upper contour sets of some house h receive equal assignment probabilities of h: for each $a, a' \in A, h \in H$ such that $U(\succ_a, h) = U(\succ_{a'}, h)$ it follows that $P_{ah} = P_{a'h}$. A random assignment P satisfies the *strong equal treatment of equals (SETE)* if any two agents with identical preferences down to some house receive an identical assignment down to that house. This can also be defined formally using upper contour sets: if two agents have identical upper contour sets of every house g down to house h, then they also receive equal assignment probabilities of h: for each $a, a' \in A, h \in H$ such that for each $g \in A : g \succcurlyeq_a h \;\; U(\succ_a, g) = U(\succ_{a'}, g)$ it follows that $P_{ah} = P_{a'h}$. The latter also implies that agents a and a' receive equal probabilities for all their houses down to h. Notice that UEF and SETE differ from the definitions of EF and ETE in that the set of agents that can be compared is different, namely, is restricted for envy-freeness and enlarged for the equal treatment of equals.

The six fairness notions introduced so far can be logically ordered: envy-freeness implies upper envy-freeness, upper envy-freeness implies the strong equal treatment of equals, which in turn implies the equal treatment of equals; weak envy-freeness and equal division lower bound are also implied by envy-freeness but neither imply nor are implied by each other and the other properties.

Remark. The following logical relations hold:

1. envy-freeness $\Longrightarrow$ upper envy-freeness $\Longrightarrow$ strong equal treatment of equals $\Longrightarrow$ equal treatment of equals;

2. envy-freeness $\Longrightarrow$ weak envy-freeness;

3. envy-freeness $\Longrightarrow$ equal division lower bound;

4. weak envy-freeness, equal division lower bound and upper envy-freeness (as well as strong equal treatment of equals and equal treatment of equals) are logically independent.

The proof of these relations can be found in the appendix.

We have now prepared all necessary definitions and their logical relations to study the first impossibility result presented in the next section.

3 First Impossibility Result

We begin by studying the tradeoff between the properties of a mechanism when fairness is of a higher concern than efficiency. The following theorem considers the set of strategy-proof mechanisms that are moderately efficient (at least ex-post efficient) and very fair (envy-free, which implies all other fairness criteria). The set of such mechanisms turns out to be empty:

Theorem 1. *For $N \geq 3$ there does not exist a mechanism that is ex-post efficient, strategy-proof, and envy-free.*

The result above is a direct corollary of a stronger result of Lemma 1:

Lemma 1. *There does not exist a mechanism that is ex-post efficient, upper-shuffle-proof, and upper-envy-free.*

Proof. We first prove the claim for $N = 3$ and we do it by contradiction. Suppose there exists a mechanism φ satisfying ExPE, USP an UEF. Consider the preference profile $\succ$ which is depicted in the following figure:

$$\succ: \begin{array}{c||ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_2 & h_1 & h_3 \end{array} .$$

At $\succ$ mechanism φ must induce the following random assignment:

$$\varphi(\succ) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} .$$

To see that let us begin with the assignment probabilities of house h_1. Agent a_3 receives zero probability $\varphi_{a_3 h_1}(\succ) = 0$ due to ExPE of φ. Agents a_1 and a_2 receive equal probabilities $\varphi_{a_1 h_1}(\succ) = \varphi_{a_2 h_1}(\succ) = \frac{1}{2}$ since φ satisfies SETE (implied by UEF), otherwise the agent who received less of her top house h_1 might have envied another agent. Next, consider the assignment probabilities of house h_2. Since agent a_2 dislikes house h_2 while agent a_3 prefers this house over others, agent a_2 is never assigned h_2 due to the ExPE of φ: $\varphi_{a_2 h_2}(\succ) = 0$. Therefore agent a_2 is left with one half of probability of house h_3: $\varphi_{a_2 h_3}(\succ) = 1 - \varphi_{a_2 h_1}(\succ) = \frac{1}{2}$. Finally, notice that the remaining assignment probability of house h_3 is spread equally between agents a_1 and a_3 due to the UEF of φ (their upper contour sets at h_3 are identical). Thus, $\varphi_{a_1 h_3}(\succ) = \varphi_{a_3 h_3}(\succ) = \frac{1}{4}$ and $\varphi_{a_1 h_2}(\succ) = \frac{1}{4}, \varphi_{a_3 h_2}(\succ) = \frac{3}{4}$.

Next consider another preference profile $\succ'$ that differs from $\succ$ in that agent a_3 copies the report of agent a_1:

$$\succ': \begin{array}{c||ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_1 & h_2 & h_3 \end{array} .$$

Then the corresponding random assignment is as follows:

$$\varphi(\succ') = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} .$$

Regarding house h_1 in this random assignment agents receive equal probability shares $\varphi_{a_1h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \frac{1}{3}$ since φ is SETE. Next, due to the ex-post efficiency of φ agent a_2 receives zero probability of being assigned house h_2 as before: $\varphi_{a_2h_2}(\succ') = 0$. Therefore we conclude that $\varphi_{a_2h_3}(\succ') = \frac{2}{3}$ and, again using SETE, $\varphi_{a_1h_2}(\succ') = \varphi_{a_3h_2}(\succ') = \frac{1}{2}$ and $\varphi_{a_1h_3}(\succ') = \varphi_{a_3h_3}(\succ') = \frac{1}{6}$.

Note that φ cannot satisfy USP since when shifting from $\succ_{a_3}$ to $\succ'_{a_3}$ the agent's a_3 upper contour set at h_3 remains the same $(U(\succ_{a_3}, h_3) = U(\succ'_{a_3}, h_3))$ but the assignment probability has changed. This contradiction completes the proof for $N = 3$.

For $N > 3$ consider the following preference profile $\succ \in \mathcal{R}^N$. Agents with indices higher than 3 prefer a house with a corresponding index to all others: $\forall a_i \in A : i > 3, \forall h \in H : h \neq h_i \implies \succ_{a_i}: h_i \succ_{a_i} h$. Additionally let the first three agents prefer the first three houses to any other house: $\forall a_i \in A : i, j = 1, 2, 3, \forall h \in H : h \neq h_j \implies \succ_{a_i}: h_j \succ_{a_i} h$. Their preferences for the first three houses are as follows:

$$
\succ: \begin{array}{c||cccccc}
a_1 & h_1 & h_2 & h_3 & ... & h_{N-1} & h_N \\
a_2 & h_1 & h_3 & h_2 & ... & h_{N-1} & h_N \\
a_3 & h_1 & h_2 & h_3 & ... & h_{N-1} & h_N \\
... & ... & ... & ... & ... & ... & ... \\
a_i & h_i & ... & ... & ... & h_{N-1} & h_N \\
... & ... & ... & ... & ... & ... & ... \\
a_N & h_N & ... & ... & ... & h_{N-2} & h_{N-1}
\end{array}.
$$

We first show that at this preference profile due to the ExPE mechanism φ assigns objects with indices higher than 3 to the corresponding agents with certainty: for each $i > 3$ $\varphi_{a_ih_i}(\succ) = 1$. Assume the opposite, namely that $\varphi_{a_jh_j} < 1$ for some $j > 3$. For φ is ExPE there must be an efficient matching μ for which $\mu(h_j) = a_k \neq a_j$. We now show the inefficiency of any such matching by constructing another matching that dominates μ. Let $ind()$ denote index function such that for each $l \leq N$ $ind(a_l) = ind(h_l) = l$. Consider the chain C of agents coupled with corresponding houses that begins with (a_j, h_j) where the next agent in the chain is the agent assigned the house of the previous couple at μ : $C = (a_j, h_j), (a_k, h_k), (\mu(h_k), h_{ind(\mu(h_k))}), ...$ If at some point in C we face one of the first three agents, then the next agent in the chain by construction must be some agent a_m with the index above three that is assigned one of the first three houses (there is at least one house among the first three which is assigned to an "outsider" with an index higher than three), $a_m : (m > 3) \cap (ind(\mu(a_m)) \leq 3)$.[10] Since N is finite and since each agent or object can appear only once in a matching, such a chain C inevitably arrives at the couple $(a_{ind(\mu(a_j))}, \mu(a_j))$ and constitutes a cycle that includes both a_j and h_j. Notice that all agents in C prefer the coupled houses to the houses assigned by μ. Therefore if they swap these houses according to C they arrive

[10]In other words we treat the first three agents and the first three houses as just one block-agent and one block-house as compared to others in order to avoid any exchanges between them. For instance, if at μ agent a_3 owns h_k, then there is some agent a_m that owns one of (a_1, a_2, a_3). After the transformation a_3 gives h_k away in exchange for this object previously owned by a_m.

at a matching that dominates μ for all agents in C which contradicts the assumption that μ is efficient and that φ is ex-post efficient.

Finally it is left to see that for the preference profile $\succ$ we can use the same arguments as for the case with only three agents as considered above to show that ExPE, USP and UEF are mutually incompatible. □

All three assumptions in the lemma are necessary. Should we drop the ExPE requirement, a uniform lottery mechanism satisfies SP and EF (and, therefore, USP and UEF). If we drop the SP requirement, then the probabilistic serial satisfies ExPE and EF (and UEF). Finally, RSD is a natural benchmark to discuss the fairness requirement. It is easy to show that RSD is always SETE because of the underlying SD procedure: the assignment probabilities for every house depend only on the preferences for the corresponding upper contour set.[11] In the same time RSD is not UEF, which is true, for instance, for the preference profile $\succ$ in the proof above. The lemma shows that this gap between SETE and UEF is so big, that even a slight compromise on strategy-proofness (requiring USP instead of SP) is not enough to close it.

Lemma 1 can be seen as a generalization of the statement in BM (p. 310) about the incompatibility of ex-post efficiency, strategy-proofness, and no envy for the case of three agents. Here we show the incompatibility of ex-post efficiency and two weaker properties: upper strategy-proofness and upper envy-freeness for any number of agents.[12]

In the following section we interchange the fairness and efficiency requirements: we relax the fairness criterion and strengthen the efficiency criterion in order to obtain a different but closely related impossibility result.

4 Second Impossibility Result

We begin by characterizing the RSD mechanism as a unique strategy-proof, ex-post efficient, and weak-envy-free mechanism for a problem with three agents.

Again, this result comes as a corollary of a more general result that involves a new fairness notion: *weak envy-freeness for equals (wEFE)*. The latter combines the properties of the two weak fairness notions: it determines whether one agent strictly prefers some other agent's assignment (like in wEF) but does so only for agents with the same preferences (like in ETE).

More formally, a random assignment P is said to be weakly envy-free for equals if for no two agents a, a' with identical preferences $\succ_a = \succ_{a'}$ one of them strictly prefers the assignment of the other: $P_{a'} >_a P_a$. A mechanism then satisfies weak envy-freeness for equals if it induces only

[11] This property is defined as a *weak invariance* in Hashimoto et. al (2014) and plays a central role in their characterization of the probabilistic serial mechanism.

[12] Perhaps BM did not show this impossibility result for the general case since they had a different focus: "For problems involving four agents and more, the impossibility result is more severe" (p.310). However, the result they show (the incompatibility of SP, OE and ETE) is logically independent from Theorem 1 and especially from Lemma 1 since ordinal efficiency is stricter than ex-post efficiency.

such random assignments. It is easy to see that weak envy-freeness for equals is implied by weak envy-freeness and by equal treatment of equals.[13]

Before we proceed, it is important to briefly mention the proving technique that is used in the proofs below. This technique usually involves *relabeling* of agents and objects in order to show the equivalence between different preference profiles. In general we are not free to relabel the agents or the houses without changing the random assignment, as that would require the mechanism to have the properties known as *anonymity* and *neutrality* respectively − none of which we assume. But if we use the properties of a mechanism (e.g., efficiency, strategy-proofness, fairness) in order to pin down specific values of an assignment probability, these properties should hold for any other preference profile of the same "type" and we can relabel agents and houses and get identical values for these probabilities for other profiles of the same "type". In other words, all the mechanism's properties that we consider are symmetrical with respect to any relabeling transformation. For instance, an ex-post efficient mechanism remains ex-post efficient regardless of any relabeling, a strategy-proof remains strategy-proof and so forth. The following Claim expresses this idea more formally:

Claim. If for some mechanism φ and some preference profile $\succ \in \mathcal{R}^N$ one can determine the value of some element in $\varphi_{ah}(\succ), a \in A, h \in H$ using the properties of φ, then this value $\varphi_{ah}(\succ)$ remains the same after any relabeling of agents and houses.

Next we use the Claim in order to restrict our attention to only six types of preference profiles (since all other preference profiles are equivalent to one of these) and pin down all the random assignment probabilities.

Proposition 1. *(Characterization of RSD) For $N = 3$ a mechanism is strategy-proof, ex-post efficient, and weakly envy-free for equals if and only if it is RSD.*

Proof. The necessity part follows from the fact that RSD is strategy-proof, ex-post efficient and satisfies equal treatment of equals. We prove the sufficiency part by checking sequentially all the preference profiles. Let φ be SP, ExPE and wEFE mechanism.

For $N = 3$ there are the following six types of preference profiles (any other preference profile can be represented as one of these after relabeling of agents and houses as discussed in the Claim above):

[13] Alternatively, one can also see the equal treatment of equals as envy-freeness for equals: if two agents have the same preferences, one of them never envies another if and only if they have identical random assignment. Therefore weak envy-freeness for equals is the weak form of this envy-freeness for equals property (similarly to the relationship between envy-freeness and weak envy-freeness).

$$\text{type 1 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_3 \succ_{a_1} h_2 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 4 (1 profile): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_2 \succ_{a_3} h_3 \end{cases},$$

$$\text{type 2 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 5 (2 profiles): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases},$$

$$\text{type 3 (1 profile): } \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_3 \succ_{a_3} h_2 \end{cases}, \quad \text{type 6 (8 profiles): } \begin{cases} h_1 \succ_{a_1} (h_2, h_3) \\ h_2 \succ_{a_2} (h_1, h_3) \\ h_3 \succ_{a_3} (h_2, h_3) \end{cases}.$$

We begin with the profile of type 1. Since φ is ExPE we get $\varphi_{a_3h_2} = 1$. Therefore agents a_1 and a_2 receive equal expected shares of the remaining houses $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_1h_2} = \varphi_{a_2h_2} = \frac{1}{2}$, otherwise one of them weakly envies another which is not allowed by weak envy-freeness for equals.

In type 2 due to the strategy-proofness agent a_2 receives the same expected share of house h_1 as before in type 1: $\varphi_{a_2h_1} = \frac{1}{2}$. Using ExPE we get $\varphi_{a_2h_2} = \varphi_{a_3h_1} = 0$ and thus $\varphi_{a_1h_1} = \varphi_{a_2h_3} = \frac{1}{2}$. Suppose also $\varphi_{a_1h_3} = x \in [0, \frac{1}{2}]$. Then the remaining probabilities are as follows: $\varphi_{a_1h_2} = \varphi_{a_3h_3} = \frac{1}{2} - x$ and $\varphi_{a_3h_2} = \frac{1}{2} + x$.

Next, consider the preference profile of type 3. Since both agents a_1 and a_2 can transform this profile to one of type 2 considered above by switching their top objects, due to SP we get: $\varphi_{a_1h_3} = \varphi_{a_2h_3} = \frac{1}{2} - x$. (Here we implicitly used the Claim above). Using wEFE for these two agents and the fact that $\varphi_{a_3h_2} = 0$ due to ExPE, we get $\varphi_{a_1h_2} = \varphi_{a_2h_2} = \frac{1}{2}$ and $\varphi_{a_1h_1} = \varphi_{a_2h_1} = x$. Consequently, the remaining expected share of house h_1 goes to agent a_3: $\varphi_{a_3h_1} = 1 - 2x$.

Finally, consider the symmetric preference profile of type 4. Each agent can swap her second and third choices and transform the preference profile to that of type 3. Due to SP their expected shares of the top house h_1 are all equal: $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_3h_1} = 1 - 2x$. Therefore $x = \frac{1}{3}$ and the random assignments of types 1−4 are identical to RSD assignments.

Now that we have determined the unknown x it is easy to show that the random assignments for the remaining profiles are also equal to RSD. □

We get two immediate corollaries from the proposition by relaxing the weak envy-freeness for equals requirement.

Corollary 1. *(BM) For $N = 3$ a mechanism is strategy-proof, ex-post efficient, and satisfies the equal treatment of equals if and only if it is RSD.*

The second corollary follows from the fact that RSD satisfies weak envy-freeness (shown in BM):

Corollary 2. *For $N = 3$ a mechanism is strategy-proof, ex-post efficient and weak envy-free if and only if it is RSD.*

Note that USP would not have been enough for the proof when moving from the type 1 profile to the type 2 and also from the type 3 to the type 4. In fact, there we use *weak invariance* (Hashimoto et al., 2014) − a "part" of strategy-proofness complementary to USP, that requires the assignment probabilities to be fixed regardless of any changes in the lower contour set. Therefore, RSD can also be characterized as a ExPE, wEFE, USP and a weakly invariant mechanism.

Next we use Corollary 2 for the second impossibility result.

Theorem 2. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and weakly envy-free.*

Proof. We prove by contradiction: assume that there exists a mechanism φ that is OE, SP and wEF.

First note that it is enough to prove the claim for the problem where $N = 4$. For the case of more agents, consider the preference profiles similar to the type used in the proof of Theorem 1, namely, where the first four agents prefer the first four houses over all other houses, and other agents prefer the corresponding house of their own index to any other house. Due to ordinal efficiency all agents with indices higher than 4 receive the corresponding houses with certainty and the assignment problem is reduced to the size of four.

For convenience of the proof we use a different notation: instead of a preference profile we use a rank table, that is a matrix $N \times N$ with rows (columns) corresponding to agents (houses) and which elements are ranks of the respective house in the preferences of a respective agent. For instance, for the preference profile $\succ^1$:

$$\begin{array}{c||cccc} a_1 & h_1 & h_2 & h_3 & h_4 \\ a_2 & h_1 & h_2 & h_3 & h_4 \\ a_3 & h_2 & h_1 & h_3 & h_4 \\ a_4 & h_4 & h_2 & h_1 & h_3 \end{array}$$

the corresponding rank table $r(\succ^1)$ is as follows (the superscripts denote the assignment probabilities):

$$r(\succ^1) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 3^{\frac{1}{3}} & 4^0 \\ 3 & 2 & 4^0 & 1^1 \end{array}.$$

For example, the rank table for agent a_3 implies that she prefers house h_2 to all others (h_2 has rank 1) and receives 2/3 of this house in expectation, she prefers house h_1 to all others besides h_2 (i.e., h_1 has rank 2) and receives zero assignment probability of h_1 and so forth.

Due to the ordinal efficiency of φ and using Corollary 2 we find that $\varphi(\succ^1) = RSD(\succ^1)$. Indeed, agent a_4 is assigned house h_4 with certainty and we can repeat the same arguments used in the proof of Proposition 1 to determine the random assignment $\varphi(\succ^1)$.

Consider now two different preference profiles $\succ^2$ and $\succ'^2$:

$$r(\succ^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 2 & 1 & 4^0 & 3 \end{array}, r(\succ'^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \end{array}.$$

Since φ is OE at $\succ^2$ and $\succ'^2$ at least two of the four agents receive zero probability of their worst houses but not necessarily all four agents (it is exactly for this reason that we need to consider two profiles and not just one). W.l.o.g. assume that these are agents a_3, a_4 for $\succ^2$ and a_2, a_4 for $\succ'^2$ (otherwise we can relabel the houses): $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$ and $\varphi_{a_2h_3}(\succ'^2) = \varphi_{a_4h_3}(\succ'^2) = 0$. We proceed with $\succ^2$ and for the profile $\succ'^2$ the argumentation line would be identical.

Now consider a preference profile $\succ^3$ that can be obtained from $\succ^2$ by changing the preferences of agent a_4 or from $\succ^1$ by changing the preferences of agent a_3 :

$$r(\succ^3) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 4^{\frac{1}{3}} & 3^0 \\ 3^0 & 2^0 & 4^0 & 1^1 \end{array}.$$

On the one hand in the random assignment of agent a_4 $\varphi_{a_4h_1}(\succ^3) = \varphi_{a_4h_2}(\succ^3) = 0$ due to ExPE of φ and $\varphi_{a_4h_3}(\succ^3) = 0$ due to SP (otherwise agent a_4 might deviate to preference profile $\succ^2$). Therefore $\varphi_{a_4h_4}(\succ^3) = 1$ and $\varphi_{a_3h_4}(\succ^3) = 0$. On the other hand in the random assignment of agent a_3 due to SP $\varphi_{a_3h_1}(\succ^3) = 0$ and $\varphi_{a_3h_2}(\succ^3) = \frac{2}{3}$ as it was at the preference profile $\succ^1$.

Next consider the preference profile $\succ^4$ obtained from $\succ^3$ but where agents a_3 and a_4 have identical preferences:

$$r(\succ^4) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Notice first that $\varphi_{a_3h_3}(\succ^4) = \frac{1}{3}$ remains the same as in $\succ^3$ due to SP. Secondly, due to ExPE $\varphi_{a_1h_4}(\succ^4) = \varphi_{a_2h_4}(\succ^4) = 0$ and $\varphi_{a_3h_1}(\succ^4) = \varphi_{a_4h_1}(\succ^4) = 0$. Thirdly, agent a_4 has to have the same random assignment as agent a_3 since their preferences are identical and we could do the same procedure where a_3 and a_4 are swapped (namely pick a_3 in $\succ^2$ and construct a profile analogous to $\succ^1$). Therefore $\varphi_{a_3h_4}(\succ^4) = \varphi_{a_4h_4}(\succ^4) = \frac{1}{2}$ and $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we are going to change the preferences of agents a_3 and a_4 sequentially so that they look symmetric to the preferences of a_1 and a_2. Consider the preference profile $\succ^5$ in which agent a_4 swaps her third and forth best houses as compared to $\succ^4$:

$$r(\succ^5) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}.$$

Note that $\varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and $\varphi_{a_4h_2}(\succ^5) = \frac{1}{6}$ due to SP and also that $\varphi_{a_1h_4}(\succ^5) = \varphi_{a_2h_4}(\succ^5) = 0$ and $\varphi_{a_3h_1}(\succ^5) = \varphi_{a_4h_1}(\succ^5) = 0$ due to ExPE. Therefore $\varphi_{a_3h_4}(\succ^5) = \varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and using wEF for a_3 and a_4 we then get that $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we do the same swap with houses h_1 and h_3 in the preferences of agent a_3 and calculate her random assignment using the same argument as above:[14]

$$r(\succ^6) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}.$$

This result is derived from the fact that $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$. But if we use the same procedure for $\succ'^2$ instead of $\succ^2$ then we get the following random assignment for a profile $\succ'^6$:

$$r(\succ'^6) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}.$$

The preference profile $\succ'^6$ is effectively identical to $\succ^6$ if we relabel houses h_1 and h_4 and agents a_1 and a_4. Due to the Claim at the beginning of this section we can conclude that agent a_2 at $\succ^6$ has to have the same random assignment as at $\succ'^6$: $\varphi_{a_2h_1}(\succ^6) = \varphi_{a_2h_4}(\succ'^6) = \frac{1}{2}$, $\varphi_{a_2h_2}(\succ^6) = \varphi_{a_2h_2}(\succ'^6) = \frac{1}{6}$ and $\varphi_{a_2h_3}(\succ^6) = \varphi_{a_2h_3}(\succ'^6) = \frac{1}{3}$. Then the full random assignment at $\succ^6$ is as follows:

[14] If in $\succ^6$ we relabel houses h_1 and h_4 and then swap agents a_1, a_2 and, on the other hand, a_3, a_4, then we get the same preference profile $\succ^6$. However, we would not be able to draw any conclusion regarding the random assignment for agents a_1 and a_2 at $\succ^6$ (agents a_3, a_4 after relabeling) since we did not determine the specific values and cannot use the logic of the Claim. For this reason we need a parallel procedure that begins with $\succ'^2$ and ends with $\succ'^6$.

$$r(\succ^6) = \begin{matrix} 1^{\frac{1}{2}} & 2^{\frac{1}{2}} & 3^0 & 4^0 \\ 1^{\frac{1}{2}} & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}.$$

Finally, agent a_2 weakly envies agent a_1 which is a contradiction. □

It is easy to see the independence of axioms in Theorem 2. First, let us weaken the ordinal efficiency requirement and demand ex-post efficiency. Then there exist at least one ex-post efficient, strategy-proof, weakly envy-free mechanism: random serial dictatorship. Next, let us drop the weak-envy-freeness requirement. Then there exists at least one strategy-proof, ordinally efficient mechanism: serial dictatorship. Finally, the probabilistic serial mechanism is an example of an ordinally efficient, (weakly) envy-free mechanism.

5 Third Impossibility Result

The last impossibility result also uses a strong notion of efficiency and a weak notion of fairness, but this time fairness is defined by the equal division lower bound.

Theorem 3. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and satisfies the equal division lower bound.*

Proof. The proof is by contradiction: assume that such a mechanism φ exists.

As before, we first prove the claim for the case $N = 4$, which can be generalized for a higher number of agents using certain preference profiles.

Consider the preference profile $\succ^1$ with the following rank table:

$$r(\succ^1) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{matrix}.$$

As before, the superscripts denote the random assignment $\varphi(\succ^1)$. Indeed, due to EDLB each agent has a right to receive at least $\frac{1}{4}$ of her most preferred house h_1 and at most $\frac{1}{4}$ of her least preferred house h_4. Then, due to ordinal efficiency, either $\varphi_{a_1h_3}(\succ^1) = \varphi_{a_2h_3}(\succ^1) = 0$ or $\varphi_{a_3h_2}(\succ^1) = \varphi_{a_4h_2}(\succ^1) = 0$ and, as it turns out, both conditions hold.

Consider now a profile $\succ^2$ that is derived from the previous profile using the swap of houses h_3 and h_4 in the preferences of agent a_1:

$$r(\succ^2) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{matrix}.$$

The random assignment $\varphi(\succ^2)$ is the same as before for the following reasons. First, the random assignment of house $h_1$is symmetric due to EDLB. Second, $\varphi_{a_1h_2}(\succ^2) = \frac{1}{2}$ because of SP (otherwise agent a_1 might deviate from/to $\succ^1$). Third, $\varphi_{a_1h_3}(\succ^2) = 0$ due to ExPE, implied by ordinal efficiency. As a result, we find the remaining element $\varphi_{a_1h_4}(\succ^2) = \frac{1}{4}$. Therefore, the random assignment of the house h_4 is again symmetric due to EDLB. Finally, using the ordinal efficiency argument we find the random assignment of houses h_2 and h_3: $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ and $\varphi_{a_3h_2}(\succ^2) = \varphi_{a_4h_2}(\succ^2) = 0$ (again: only one of these conditions has to be satisfied due to OE, but in fact both of them hold because of the previous findings).

Next, consider the preference profile $\succ^3$ derived using the same swap of houses h_3 and h_4 but this time for agent a_2:

$$r(\succ^3) = \begin{array}{cccc} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{array}.$$

It turns out that the random assignment is again the same. First, $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ due to ExPE. Second, both $\varphi_{a_1h_2}(\succ^3)$ and $\varphi_{a_2h_2}(\succ^3)$ are equal to $\frac{1}{2}$ because of SP (otherwise one of the two agents a_1,a_2 would have switched from/to preference profile $\succ^2$). The rest of the random assignment can be found using EDLB as before.

Next we consider a different preference profile $\succ^4$ in which the agents have opposite tastes regarding the other pair of houses: h_3 and h_4 (and not h_2 and h_3 as before):

$$r(\succ^4) = \begin{array}{cccc} 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4^0 & 3^{\frac{1}{2}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4^0 & 3^{\frac{1}{2}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{2}} & 4^0 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{2}} & 4^0 \end{array}.$$

The random assignment $\varphi(\succ^4)$ can be determined using the same argumentation line as in the case of $\succ^1$.

Finally, we consider the preference profile $\succ^5$, which can be derived from the profile $\succ^4$ using a swap of houses h_2, h_3 in the preferences of agent a_4, *and in the same time* from the profile $\succ^3$ using the swap of houses h_2, h_3 in the preferences of agent a_3:

$$r(\succ^5) = \begin{array}{cccc} 1^{\frac{1}{4}} & 2 & 4^0 & 3 \\ 1^{\frac{1}{4}} & 2 & 4^0 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{4}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{3}{4}} & 4^0 \end{array}.$$

The random assignment $\varphi(\succ^5)$ can be determined using the following arguments. First, since φ is SP, the elements $\varphi_{a_3h_4}(\succ^5)$ and $\varphi_{a_4h_4}(\succ^5)$ must correspond to the elements of $\varphi(\succ^3)$ and $\varphi(\succ^4)$ respectively: $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_4h_4}(\succ^5) = 0$. Second, we apply the ordinal efficiency argument to houses h_3 and h_4 and find that since $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4} > 0$, the corresponding probabilities of agents a_1, a_2 are zero: $\varphi_{a_1h_3}(\succ^5) = \varphi_{a_2h_3}(\succ^5) = 0$. Third, due to ExPE $\varphi_{a_4h_2}(\succ^5) = 0$. Fourth, the assignment of house h_1 is identical due to EDLB. Therefore $\varphi_{a_4h_3}(\succ^5) = \frac{3}{4}$ and then $\varphi_{a_3h_3}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_3h_2}(\succ^5) = \frac{1}{4}$.

So far there is no contradiction with our assumptions. However, the fact that $\varphi_{a_3h_2}(\succ^5)$ equals $\frac{1}{4}$ and is therefore different from $\varphi_{a_1h_2}(\succ^5)$ or $\varphi_{a_2h_2}(\succ^5)$ (since their sum has to be equal to one) contradicts the strategy-proofness of φ. Indeed, consider the profile $\succ^6$ which is different from $\succ^5$ in that agent a_3 swaps her preferences for houses h_3 and h_4 and thus becomes identical to agents a_1 and a_2:

$$r(\succ^6) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 3^{\frac{1}{4}} & 2 & 4 \end{matrix}.$$

Since any of the agents a_1, a_2, a_3 could swap their least preferred houses h_3, h_4 in order to deviate from/to $\succ^5$, due to strategy-proofness of φ we conclude that $\varphi_{a_1h_2}(\succ^6) = \varphi_{a_2h_2}(\succ^6) = \varphi_{a_3h_2}(\succ^6) = \frac{1}{4}$ and therefore $\varphi_{a_4h_2}(\succ^5) = \frac{1}{4}$ which contradicts the ex-post efficiency of φ. □

An important implication of Theorem 3 is the restriction that it puts on the feasibility set of mechanisms that dominate RSD. Notice first that RSD satisfies the equal division lower bound. Indeed, in the RSD procedure each agent has an equal chance to be the first in the ordering (and thus receive her first best house), the second (and thus receive at least her second best) and so on. Therefore, under the RSD assignments all agents are weakly better off than under the uniform lottery.

Corollary 3. *For $N > 3$ any ordinally efficient mechanism that dominates RSD is not strategy-proof.*

The corollary, however, does not restrict the set of mechanisms that dominate RSD *without* being ordinally efficient. Thus, in the set of strategy-proof mechanisms there might still be room for improvement upon RSD.

6 Conclusions

This paper considers the standard random assignment problem of assigning N indivisible objects to N agents and shows the impossibility for a strategy-proof mechanism to be simultaneously fair and efficient (in three specific ways). Theorem 1 shows the impossibility to combine a weak notion of efficiency − ex-post efficiency, with a strong notion of fairness − envy-freeness; it is the first known impossibility result in the related literature that involves ex-post efficiency. Theorem 2 shows the impossibility for the opposite set of properties: a weak notion of fairness − weak envy-freeness and a strong notion of efficiency − ordinal efficiency. I also show that for the case of three agents the trinity of strategy-proofness, ex-post efficiency, and weak envy-freeness for agents with identical preferences uniquely defines the random serial dictatorship mechanism. Finally, Theorem

3 shows a similar impossibility result with a different weak fairness notion: equal division lower bound.

The first theorem is, perhaps, of the highest importance for the practical implementation of matching and random assignment mechanisms since it deals with the commonly required properties of strategy-proofness and ex-post efficiency. The other two theorems resemble the impossibility result of Bogomolnaja and Moulin (2001), although with arguably more relevant notions of fairness.

All three results in this paper can also be considered as a support for using RSD in random assignment problems when strategy-proofness is of high importance. As demonstrated in the Introduction, strategy-proofness, ex-post efficiency, and weak envy-freeness are strongly desirable properties for a mechanism used in real-life applications, while the equal division lower bound might be important when switching from one mechanism to another. Not only does RSD possess all four of these properties, but, as this paper demonstrates, it is also impossible to improve on any of the weak properties: to demand ordinal efficiency instead of ex-post efficiency, or envy-freeness instead of weak envy-freeness.

Moreover, when there are only three agents RSD is the unique strategy-proof and ex-post efficient mechanism that satisfies the weakest (among presented here) fairness notion: weak envy-freeness for equals. It, however, remains unclear, what combination of properties characterizes RSD for the general case. The characterization result in this paper cannot be directly generalized even for the case of four agents (however, there are also no counter examples found). The reason for this is that weak envy-freeness (and especially weak envy-freeness for equals) is not handy enough as compared to the equal treatment of equals. For instance, for two agents with identical preferences weak envy-freeness gives precise implications only in case these agents receive identical probabilities for all but two objects. Then the two agents have to have the same random assignment for the remaining objects as well. Equal treatment of equals, on the contrary, has implications for the assignment probabilities of all objects. Therefore, I believe, generalizing this characterization result would be more difficult than the result which uses equal treatment of equals.

Another open question is to what extent can one of the three properties be satisfied should the other two be taken at their extreme. For instance, if ordinal efficiency and envy-freeness are satisfied, then the probabilistic serial mechanism appears to be the "most" strategy-proof mechanism since it is weakly invariant (limits the set of profitable deviations) and weakly strategy-proof (which means that no agent can receive a stochastically dominant assignment by manipulating). Similarly, one could be interested in the "most fair" mechanism that satisfies strategy-proofness and ordinal efficiency (since the only known SD mechanism is very unfair), and in the "most" efficient mechanism that satisfies strategy-proofness and envy-freeness (again, the only known equal division or pure lottery mechanism disregards preferences and therefore is almost always inefficient).

It should also be mentioned that some of the results of this paper are limited by the nature of the standard framework that is used. In a more general setting where the number of houses is higher than the number of agents, especially in the case with an outside option or a null object, the agents have a richer strategy set and thus the results cannot be directly transferred to that setting. For instance, in such settings RSD loses ex-post efficiency and can be dominated by a strategy-proof mechanism (see Erdil (2014) for these and other results in the general setting). However, the main negative results must hold since the standard setting is the special case of the general setting.

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A Appendix

Proof of the Remark in section 2.

Proof. Envy-freeness$\Longrightarrow$ upper envy-freeness. We need to show that for each envy-free random assignment P it follows that for each $a, a' \in A$ and each $h \in H$ if $U(\succ_a, h) = U(\succ_{a'}, h)$ then $P_{ah} = P_{a'h}$. First note that $F(\succ_a, h, P_a) = F(\succ_{a'}, h, P_{a'})$ since otherwise one of the two agents might envy another (e.g., if she is almost indifferent between all objects in her upper contour set of h). Then notice that $F(\succ_a, h_a, P_a) = F(\succ_{a'}, h_{a'}, P_{a'})$ where h_a and $h_{a'}$ are the least preferred objects in $U(\succ_a, h)\backslash\{h\}$ and $U(\succ_{a'}, h)\backslash\{h\}$ respectively − for the same reason as earlier. Finally $P_{ah} = F(\succ_a, h, P_a) - F(\succ_a, h_a, P_a)$ and $P_{a'h} = F(\succ_{a'}, h, P_{a'}) - F(\succ_{a'}, h_{a'}, P_{a'})$ which completes the proof.

Upper-envy-freeness $\Longrightarrow$ strong equal treatment of equals. Here we need to show that for each upper envy-free random assignment P it follows that for each $a, a' \in A$ with identical preferences down to some $h \in H$ the random assignment down to this h is the same or, more formally, for each $h' \in H$ such that $h' \succ_a h$ and $h' \succ_{a'} h$ it follows that$P_{ah'} = P_{a'h'}$. We prove by induction: consider the top object $h_1 : h_1 \succ_a h'$ for each $h' \in H$ (and $h_1 \succ_{a'} h'$ since the preferences down to h are identical). Using the upper envy-freeness for h_1 (since $U(\succ_a, h_1) = U(\succ_{a'}, h_1)$) we get $P_{ah_1} = P_{a'h_1}$. We then do it for the second top object and so forth until we reach h which would complete the proof.

Strong equal treatment of equals $\Longrightarrow$ equal treatment of equals. For ETE we need to consider only agents with identical preferences. Clearly, for any two of these agents the strong equal treatment of equals implies equal treatment of equals since SETE applies to all objects.

Envy-freeness $\Longrightarrow$ weak envy-freeness. This is true since if agents prefer their own assignments, then none of them strictly prefers the assignment of someone else.

Envy-freeness $\Longrightarrow$ equal division lower bound. Consider some agent $a \in A$ and her top object $h_1 \in H$. Since the assignment P is envy-free there is no agent a' with $P_{a'h_1} > P_{ah_1}$ (otherwise a could possibly envy a'). Therefore agent a gets at least her fair share of object h_1 of $\frac{1}{N}$. Next, consider the two top objects $\{h_1, h_2\}$ of agent a. Similarly, there is no agent a' with the total probability $(P_{a'h_1} + P_{a'h_2})$ higher than the total probability of agent a for the same two objects (otherwise a would envy a' once she is indifferent between h_1 and h_2 and does not care as much about the rest). Therefore the total probability $(P_{ah_1} + P_{ah_2})$ is at least as high as the fair share $\frac{2}{N}$. We use the same logic for the other objects and find that agent a is weakly better off under P than under the equal division.

Independence of properties. Finally, it is left to show the mutual independence of the weak notions of fairness which is fairly easy to do by a contour example for each two notions. Indeed, these examples are easy to come up with since all the notions have a different nature: UEF, SETE and ETE can be applied to those preference profiles in which some preferences are (partially) identical; wEF and EDLB apply to all preference profiles but wEF compares assignments between different agents, while EDLB compares them to the fair division. □